Proof: The foregoing results show that $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ satisfies (*i*), (*ii*), (*iii*). Hence, by the proof of the first structure theorem, every ideal in $\mathfrak{J}'$ has a complementary ideal. In particular, $\mathfrak{J}' = \mathfrak{J} \oplus \mathfrak{B}$, where $\mathfrak{B}$ is an ideal. Then $\mathfrak{J}$ has an identity element.

Axioms (*iii*) and (*iv*) are evidently satisfied if $\mathfrak{J}$ is finite dimensional over Φ. Also, it is quite easy to show in this case that (*ii*) is equivalent to the assumption that $\mathfrak{J}$ has no nil ideals. Therefore, our results give a new and improved derivation of Albert's structure theorems for finite dimensional semisimple Jordan algebras.

[1] See author's paper, "A coordinatization theorem for Jordan algebras," these PROCEEDINGS, **48,** 1154–1160 (1962); and the references in this paper.

[2] See ref. 1; also Jacobson, N., "A theorem on the structure of Jordan algebras," these PROCEEDINGS, **42,** 140–147 (1956); and McCrimmon, K., "Norms and non-commutative Jordan algebras," forthcoming in *Pacific J. Math.*

[3] "Jordan algebras of self-adjoint operators," *Mem. Am. Math. Soc.*, **53** (1965).

[4] See ref. 1, p. 1154, and the reference given there to a dissertation by Dallas Sasser.

[5] See ref. 1, p. 1158.

[6] "Simple alternative rings," *Ann. Math.*, **58,** 544–547 (1953).

[7] "Lie and Jordan systems in simple rings with involution," *Am. J. Math.*, **78,** 629–649 (1956).

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*FUNDAMENTAL POLYHEDRONS AND LIMIT POINT SETS OF KLEINIAN GROUPS**

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1. We denote by Ω the group of orientation-preserving Möbius transformations in R^3 which leave $B = \{x/|x| < 1\}$ invariant. In other words, transformations in Ω are products of an even number of reflections in spheres orthogonal to the unit sphere $S = \partial B$.

Any discrete subgroup of Ω is called a Kleinian group. Poincaré[1] has shown that any such group G is discontinuous in B. In its action on S it is discontinuous on an open set D, which may be empty. The complement $L = S - D$ is the set of limit points. If L is all of S, we say that G is of the first kind. If not, L is nowhere dense on S and G is of the second kind. This terminology stresses the analogy with Fuchsian groups.

2. The orbit space $M = (B \cup D)/G$ is a connected orientable 3-manifold with boundary. The latter can be identified with $M_0 = D/G$ which need not be connected, but whose components carry a structure of Riemann surface.

The immediate problem is to study the structure of M and M_0 as well as the properties of D and L, particularly when G is a finitely generated group. So far only M_0 has been investigated with some degree of success.[2] It is hoped that a systematic study of M will lead to more complete results, also as far as M_0 and L are concerned.

3. For the study of M it is advantageous to introduce the isometric funda-

mental polyhedron, already considered by Poincaré. It is defined only when the origin is not a fixpoint, but this is no serious restriction since we can always replace G by a conjugate subgroup with this property.

The transformations in Ω will be denoted $x \rightarrow Ax$. The linear ratio $|dAx|:|dx|$ is independent of the direction, and to stress the analogy with the plane case it will be denoted by $|A'(x)|$. We observe that $|A'(x)| = 1$ on a sphere K_A orthogonal to S, namely, the noneuclidean plane whose points are equidistant from 0 and $A^{-1}0$. Indeed, if A_0 denotes reflection in K_A, then A_0A^{-1} leaves 0 fixed and is hence an isometry; it follows that $|A'| = |A_0'| = 1$ on K_A. Since $|A_0'| > 1$ inside and $|A_0'| < 1$ outside of K_A, the same reasoning shows that $|A'(x)| > 1$ inside and $|A'(x)| < 1$ outside of K_A.

We call K_A the *isometric sphere* of A and its intersection with S the *isometric circle*. The transformation A maps K_A on $K_{A^{-1}}$, and this mapping is a euclidean congruence. We point out that the isometric circles are defined by means of the spherical derivative and are therefore not identical with the isometric circles of Ford.[3]

DEFINITION. *The isometric polyhedron of G is the set P of all $x \in B \cup D$ such that $|A'(x)| < 1$ for all $A \in G$ except the identity.*

In other words, P is the intersection of the outsides of the isometric spheres K_A, $A \in G$. These spheres accumulate only toward L, as seen from the fact that $A^{-1}0$ lies inside K_A. Thus P can be described as a convex noneuclidean polyhedron such that any compact set in $B \cup D$ meets only a finite number of its sides and edges.

If K_A contains a side of P, so does $K_{A^{-1}}$. These sides are equivalent under the mapping A, and at the same time congruent in the euclidean and noneuclidean sense.

The intersection $P_0 = P \cup D$ is a fundamental polygon for G acting on D. Its sides are circular arcs, but there is no convexity, and P_0 need not be connected. The sides are congruent in pairs.

The manifold $M = (B \cup D)/G$ and its boundary $M_0 = D/G$ can be constructed by identifying corresponding sides, edges, and vertices of P and P_0.

4. The situation becomes particularly simple if P has only a finite number of sides. In that case, G is finitely generated, namely, by the transformations that map corresponding sides on each other. The hope of proving the converse has been shattered by a counterexample due to L. Greenberg (unpublished).

In this paper we shall prove:

THEOREM. *If P has a finite number of sides, then either L is all of S, or the areal measure of L is zero.*

It is conceivable that mes $L = 0$ for all finitely generated groups, but we are unable to prove or disprove this conjecture.

5. The proof makes decisive use of the hyperbolic metric $ds = |dx|/(1 - |x|^2)$. We recall that the second Beltrami operator corresponding to this metric is given by

$$\Delta_2 u = (1 - r^2)^2 \left(\Delta u + \frac{2r}{1 - r^2} \frac{\partial u}{\partial r}\right),$$

where Δ is the ordinary Laplacian. A function which satisfies $\Delta_2 u = 0$ may be called hyperbolically harmonic. In contrast to ordinary harmonic functions in space, a hyperbolically harmonic function remains such when composed with an

$A \in \Omega$. This explains the importance of this class for the problem under consideration.

The Poisson formula for hyperbolically harmonic functions reads

$$u(x) = \frac{1}{4\pi} \iint_{|y|=1} \left(\frac{1 - |x|^2}{|x - y|^2}\right)^2 u(y)d\sigma(y),$$

where $d\sigma$ is the area element. For our purposes we choose $u = 0$ on D, $u = 1$ on L, that is,

$$u(x) = \frac{1}{4\pi} \iint_L \left(\frac{1 - |x|^2}{|x - y|^2}\right)^2 d\sigma(y).$$

If mes $L \neq 0, 4\pi$, as we shall now assume, this function is not a constant. It satisfies $u(Ax) = u(x)$ for all $A \in G$.

6. Denote by P_r the part of P in $|x| < r$ and by θ_r the part on $|x| = r$. The area of θ_r will be denoted by $r^2\theta(r)$ so that $\theta(r)$ is the solid angle subtended by θ_r at the origin.

Green's formula yields

$$V(r) = \iiint_{P_r} \text{grad}^2\, u \frac{dx}{1 - |x|^2} = \iint_{\partial P_r} u\frac{\partial u}{\partial n} \frac{d\sigma}{1 - |x|^2}. \tag{1}$$

But u has equal and $(\partial u/\partial n)$ opposite values at equivalent boundary points. Therefore, the formula reduces to

$$V(r) = \iint_{\theta_r} u\frac{\partial u}{\partial r} \frac{r^2 d\omega}{1 - r^2}, \tag{2}$$

where we have written $d\omega$ for the element of solid angle.

We shall also set

$$m(r) = \iint_{\theta_r} u^2 d\omega. \tag{3}$$

From (1), (2), and (3) we obtain at once

$$V(r)^2 \leqq \frac{r^2}{1 - r^2} m(r)V'(r) < \frac{1}{1 - r} m(r)V'(r),$$

and hence

$$\int_{r_0}^{1} \frac{1 - r}{m(r)} dr < \int_{r_0}^{1} \frac{dV(r)}{V(r)^2} \leqq \frac{1}{V(r_0)}. \tag{4}$$

Consider the equation

$$m(r) - \theta(r) = \iint_{\theta_r} (u^2 - 1)d\omega.$$

Since θ_r shrinks with increasing r and $u^2 - 1 \leqq 0$, we may conclude that

$$m'(r) - \theta'(r) \geqq 2 \iint_{\theta_r} u\frac{\partial u}{\partial r} d\omega > 0.$$

Hence, $m(r) - \theta(r)$ is increasing. If the integrand in (3) is written as χu^2 where χ is the characteristic function of θ_r, it becomes clear that the integrand tends to 0 except on radii which end on the boundary of P_0 or at a finite number of cusps. In view of the boundedness it follows that $m(r) \to 0$ for $r \to 1$. On the other hand, $\theta(r)$ decreases to $\theta(1)$, the area of P_0.

We conclude that $m(r) \leqq \theta(r) - \theta(1)$. But for a finite polyhedron it is geometrically evident that $\theta(r) - \theta(1) = O((1 - r)^2)$. This makes the integral

$$\int_{r_0}^{1} \frac{1 - r}{m(r)} dr$$

divergent, contrary to (4). We have thus proved that the measure of L is either 0 or 4π.

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[1] Poincaré, H., "Mémoire sur les groupes kleinéens," *Acta Math.*, **3** (1883).

[2] Ahlfors, L., "Finitely generated Kleinian groups," *Am. J. Math.*, **86**, No. 2 (1964), pp. 413–429.

[3] Ford, L. R., *Automorphic Functions* (New York: Chelsea Publishing Co., 1951), 2nd ed.

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*INVESTIGATION ON GROUPS OF EVEN ORDER, II**

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1. *Results on Groups G with a Given 2-Sylow Subgroup.*—(*a*) If p is a prime and P a given p-group, we can propose to study the finite groups G which possess P as their p-Sylow subgroup. The theory of blocks of characters[1] can be applied. In I, results concerning the irreducible characters of G were obtained. As was shown in II, additional methods are available for $p = 2$. The nature of the results depends strongly on the structure on P. At least for certain P, an amazing amount can be said about G in the case $p = 2$. Elsewhere, the cases of quaternion dihedral, quasidihedral P have been investigated and at least partial results have been obtained for abelian P. In this section, we shall report on some further results of this nature.

(*b*) The problem of characterizing the projective groups of Desarguesian planes of order q for $q \equiv 1 \pmod 4$ requires the investigation of groups P with generators σ_1, σ_2, τ defined by the relations

$$\sigma_1^{2^m} = \sigma_2^{2^m} = 1, \qquad \tau^2 = 1; \qquad \sigma_1\sigma_2 = \sigma_2\sigma_1, \qquad \tau^{-1}\sigma_1\tau = \sigma_2.$$

Here, m is an integer; $m \geqq 2$. Thus, P has order 2^{2m+1}, and P is the wreath product of a cyclic group of order 2^m by a group of order 2. We use the following notation,

$$J_1 = \sigma_1^{2^{m-1}}, \quad J_2 = \sigma_2^{2^{m-1}}, \quad J = J_1J_2, \quad \alpha = \sigma_1\sigma_2, \quad \beta = \sigma_1\sigma_2^{-1}, \quad S = \langle\sigma_1,\sigma_2\rangle.$$